Phases or Proxels: The Decision Factors

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Abstract. Proxel-based simulation is a new and deterministic approach to analysing discrete stochastic models and studying their behaviour. One of its main advantages is the intuitiveness with which it approaches models' analysis. Even though the proxel-based method is a deterministic approach, unlike the other approaches in that class, it does not set up and solve partial differential equations, and possesses intuitiveness which is comparable to the one that characterises discrete-event simulation. Unfortunately, as it is also the case with other deterministic approaches it suffers from the well-known state-space explosion problem, prohibiting its application on large-scale models.

One of the ways to reduce the problem of state-space explosion is the inclusion of discrete phase-type approximations, which is possible because of the fact that the underlying stochastic process of the proxel-based method is a discrete-time Markov chain, in which the probabilities for staying in the same state are zero. The use of phases is not always recommendable, and sometimes their approximation can be very expensive in terms of computation time, so that it more than compensates for the saved proxel simulation time. Therefore, we need a way to decide when it is a good decision to substitute a distribution by a particular phase-type approximation, and when proxels might be the better choice.

In this paper we use our recently introduced paradigm of a lifetime of a discrete state as another decision indicator, besides the distributions characteristics, when choosing between phases and proxels. The whole decision making process is supported by a demonstrative example.

1 Goals of the Paper

The main goal of this paper is to strengthen the decision of whether to substitute a general distribution by a discrete phase-type one for the proxel-based simulation of a given model. Until now this decision was made based on the support of the probability density function of every state change individually, independent of the characteristics of the competing state changes, and its coefficient of variation, as implemented in the tool described in [IH05b]. The approach, however, can be extended to include other factors as well in order to increase the efficiency of the proxel-based simulation combined with discrete phases. One of the most relevant factors is the lifetime of the corresponding discrete state, which is introduced in [LMH05b] and determines the longest time that the model can reside in a given discrete state, based on the simulation parameters. The meaning of this extension of the decision factors is discussed further in the paper.

2 Introduction

In this section we provide a brief overview of the proxel-based method and its combination with discrete phase-type distributions.

2.1 Brief Overview of the Proxel-Based Method

The proxel-based method implements the method of supplementary variables [Ger00,Cox55] and works by observing all of the possible behaviours of the model. Every possible development has determined computable probability, based on the distribution functions which describe the events that cause the state changes, as well as the times during which they have been pending (denoted as age intensities). The unit that stores all necessary information for turning a non-Markovian model into a Markovian one is referred to as *proxel*, which stands for "probability element". Among other parameters, it contains the discrete state, the age intensities of the possible events in that discrete state, and the probability, resulting in the following structure:

$$Proxel = (State, Time, Route, Probability), \text{ where}$$
$$State = (Discrete\ State, Age\ Intensity\ Vector).$$

The *Route* parameter contains the sequence of states via which the model has reached the actual state. Time advances in discrete steps and the point where the simulation starts is the initial discrete state. From there on, based on the possible state changes (associated with events) new proxels are generated for the subsequent time steps. The values of the corresponding age intensities are updated with respect to the event that has caused the state change.

The *probability* is approximated by the IRF (instantaneous rate function) $\mu(\tau)$ [Tri02], integrated along the time step, where τ is the age intensity of the active state change i.e.

$$probability = \int_t^{t+\Delta t} \mu(x)dx, \text{ approximated by } probability = \mu(t) \times \Delta t, \qquad (1)$$

which in this case we interpret as the probability that the state change has happened within the interval $[t, t + \Delta t)$. The IRF is computed from the distribution functions (CDF and PDF) as shown in [Tri02]. More on the proxel-based method can be found in [Hor02,LMH05a,LM05].

2.2 Combining Phases and Proxels

Phase-type distribution functions can be very efficient and accurate when approximating for smooth distribution functions with infinite support, such as Weibull or Normal. On the other hand, they are very expensive to fit when approximating finite support distributions like Uniform and Deterministic which usually require large number of phases to be approximated more accurately.

Above described features of the discrete phase-type approximations yielded another variation of the proxel-based method, which is achieved by its combination with discrete phases, as described in [ILMH05]. The basis for the combination is the fact that the underlying model of the proxel-based simulation can be described as a DTMC which is also what the discrete phases are. Therefore, the implementation of the discrete phases for substituting some of the distributions fitted well into the existing framework. An illustrative example is a model which consists of two state changes distributed according to a Weibull and a Uniform distributions correspondingly. The uniformly distributed state change is very expensive to approximate using discrete phase-type distribution functions and is therefore simulated with proxels. By contrast, in most cases, depending on the parameters a Weibull can be sufficiently well approximated by using only a few phases.

When combining proxels and phases, a new element is added to the proxel i.e. the phase (or phases) in which the model is, thus extending the definition of a proxel. The extended proxel structure is the following:

$$\begin{gathered} Proxel = (State, Time, Route, Probability), \text{ where} \\ State = (Discrete\ State, Age\ Intensity\ Vector, Phase\ Vector). \end{gathered}$$

The *Phase Vector* contains information about the current phases of the model, in the same way as the *Age Intensity Vector* carries information about the relevant age intensities. The other elements have the same meanings as explained in the previous section.

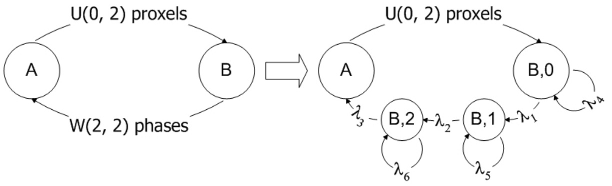

Fig. 1. Inclusion of phases in a simple two state model

In Figure 1 an example model is shown for the purpose of illustrating the basic idea of the approach of combining discrete phases with proxels. The model consists of two discrete states, and two state changes, one of which is Weibully distributed and can be accurately fitted using discrete phases. In the same figure, the model development is shown after the inclusion of phases. The state change from B to A is approximated using three phases, whereas the state change from A to B is left unchanged because of the fact that proxels are more suitable for approximating Uniform distribution. When the model is in state A, the supplementary variable is the age intensity, and whenever the model is in state B, the supplementary variable denotes the phase of the discrete state B. The values of λ_i, which are now probabilities of the corresponding phase changes, can be computed using one of the fitting algorithms described in [IH05a] and they do not require age variables because of the property of memorylessness. The state development for the first five time steps is shown in Figure 2, in which the variable supplementing the discrete state A denotes an age intensity, and the one supplementing the discrete state B denotes a phase.

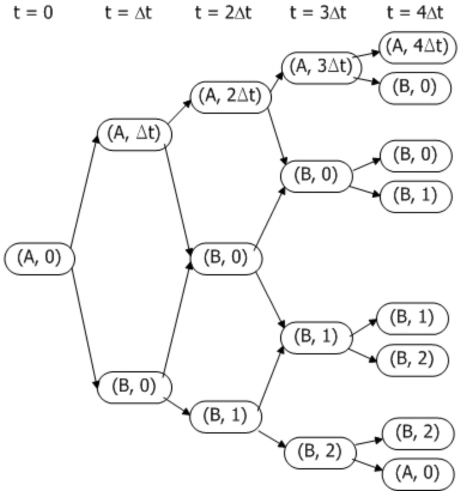

Fig. 2. Proxel-phases state tree

The concept of phases fits well in the existing framework because the underlying processes of both approaches are DTMCs. This results in a straightforward implementation of the combined approach, for which recently a tool has been introduced [IH05b]. The basic algorithm of this approach of combining phases and proxels is presented in [ILMH05].

3 Phases or Proxels: The Decision Criteria

The decision criteria for substituting proxels by phases consists of two main factors: characteristics of the distributions associated with the state changes and lifetimes of the discrete states in a given model. Both factors are described in the following.

3.1 Characteristics of the Distributions

The main goal in the decision process is to choose between a proxel or a phase-type approximation of a general distribution function. This decision is made based on an estimate whether the runtime of the approximation will outweigh the benefit of replacing a general distribution function by a discrete phase-type approximation. Until now this process is guided solely by certain distribution characteristics and the discretisation time step. Both help estimate the runtime of the approximation algorithm with certain optimization methods and phase numbers.

A stochastic distribution function can be characterized in a number of ways. The type of distribution function and its parameters are only a few. Other common characteristics are mean, variance, moments and other distribution specific ones.

An obvious decision factor is the type of distribution. Finite support distribution functions or ones that have jumps in the PDF (probability density function), such as Deterministic or Uniform distributions, should not be replaced by phase-type approximations. To achieve an accurate result, the approximations would need a large number of phases and take long to compute.

One distribution characteristic that foremost influences the fitting process is the coefficient of variation (CV), which is defined as the standard deviation over the mean, and therefore is a measure of the relative variance of a distribution. The larger the CV of a distribution, the fewer phases are needed to approximate it accurately, which leads to lower computation cost. This is mostly due to a small or non-existent period of steps with zero probability in the first section of the distribution. When using a Weibull distribution, the shape parameter is also indicator, where a smaller shape parameter results in a larger CV, and therefore fewer phases are needed for an accurate approximation.

The third factor that influences the decision whether to replace a distribution is the number of approximation time steps. This is determined by the discretisation time step and the support of the distribution function or the cut-off point for the discretisation. It is only feasible to replace a distribution if the number of discretization time steps is within certain bounds (10 to ≈ 5000). Then they can produce accurate approximations in reasonable computation time. If the distribution has less than 10 discretization time steps, replacing it by a phase type approximation would produce unnecessary overhead for the already small representation. If the number of time steps is too large, the computation time of the approximation becomes infeasible and for small phase numbers the results are not accurate enough.

The coefficient of variation is also used to determine the number of phases and optimization method that should be used for the approximation. The other factor that influence that decision is the mean/Δt (Weibull scale/Δt). Both of these combined form a decision tree like scheme to determine the preferred optimization method and necessary number of phases that do not lead to an excess of approximation time.

3.2 Lifetimes of Discrete States

The lifetime of one discrete state of a given model determines the longest time that the model can reside in that discrete state. The lifetime parameter is dependent on the concrete simulation of a model and the minimum probability threshold, which is why it cannot be attached as a property to the model itself. The algorithm and the complete definition of the lifetime of a discrete state is provided in [LMH05b]. The lifetimes of all discrete states directly determine the computational complexity of the model. The computation of lifetimes is straightforward when finite support distributions are involved. However, in the other cases it requires a minimum probability threshold ϵ which determines the truncation point, which needs to be defined such that no significant probability values get ignored.

The meaning of the lifetimes regarding the decision of substituting general distributions by discrete phase-type ones is that it can happen that the lifetime is significantly shorter than the support of the corresponding distribution that it can result into unneeded expense for performing approximation. This case is illustrated in Figure 3, which refers to the model shown in Figure 4, where the lifetime of the discrete state A is significantly shorter than the support of the distribution of the state change AC. Additionally, as shown further in the experiments, the complexity of the simulation

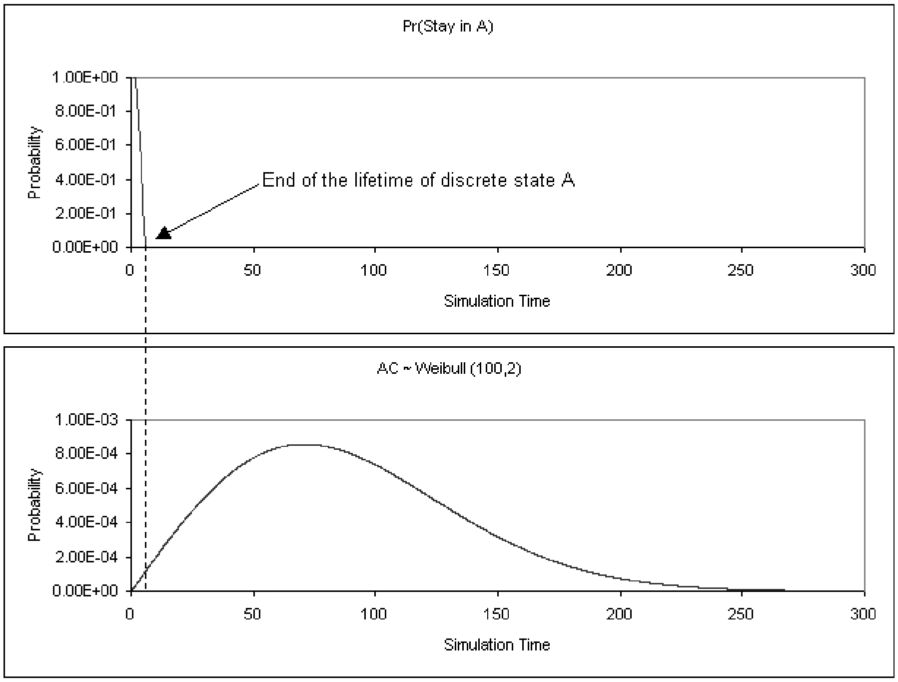

Fig. 3. Probability for staying in the discrete state A and the PDF of the state change AC

can be increased too. That is especially if the substitution is an expensive one i.e. one that requires many phases and long fitting times, in which case the savings can be significant.

4 Numerical Example

In this section we present a simple model for the purpose of demonstrating the effect that the calculation of lifetimes can have on the decision for substituting proxels by phases, and thereby on the efficiency of the simulation. The experiments are performed using the tool presented in [IH05b].

The example model presented by the state-transition diagram in Figure 4 consists of three discrete states: A, B and C, and four state changes with enabling memory policy: AB, AC, BC and CB, distributed according to Weibull and Uniform distributions. We choose $\Delta t = 0.1$ and $\epsilon = 10^{-15}$. The model is simulated up to time $t = 100$. Correspondingly, the lifetimes of the three discrete states are the following:

- lifetime(A) = $5 \times \Delta t$,
- lifetime(B) = $10 \times \Delta t$, and
- lifetime(C) = $150 \times \Delta t$.

If we ignore the lifetimes, the following state changes will be approximated by phases: AC and CB, as is also suggested by the phase approximation tool. The tool

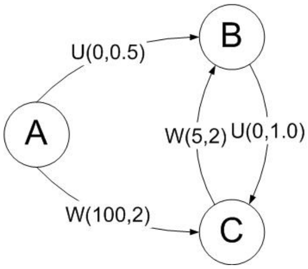

Fig. 4. Example model

also suggests that both of the approximations use 6 phases for achieving an acceptable approximation. The computation times in that case, both for the fitting and the proxel-based simulation are shown in Table 1 and denoted as *Case Ph*.

	Computation Time (*Case Ph*)	Computation Time B (*Case Prox*)
Fitting Time	0.4(6ph) s + 4.9(6ph) s	0 s
Simulation Time	2.95 s	0.36 s
Total Time	8.25 s	0.36 s
Number of Proxels	1418880	217623

Table 1. Computation times for the two simulation cases: excluding and including the effect of the lifetimes

In the second case (*Case Prox*) the decision about the type of approximation is made considering the lifetimes of the discrete states, i.e. as an intersection with the decision made based on the distribution characteristics. For this example model that would mean no approximations because of the short lifetimes of the discrete states in which the approximation-suggested state changes are active. The results of this experiments are also shown in Table 1, referred to as *Case Prox*. The total computation time needed for the phase-proxel simulation is more than 20 times as large as the one the standard proxel-based implementation. It is especially interesting that the phase-proxel simulation generates a larger number of proxels by a factor of ca. 7. This shows that for the model described in Figure 4 the inclusion of phases adds to the complexity of the simulation too. The reason for this behaviour is the feature of the proxel-based method to store only the truly reachable states of a model, whereas the phase-proxel variant generates and stores all of the phases that are computed with the approximation algorithm. Additionally, an important argument for the standard proxel algorithm is that it approximates the distributions more accurately, as stated in [IH05a].

5 Summary and Outlook

In this paper we extend the list of criteria involved in the decision process when approximating general distributions by phase-type ones with the goal of improving efficiency of the proxel-based simulation. The new factor is the lifetime of the discrete state in which the state change is active. It provides additional information about the actual support of the distribution of that state change. The current set of decision criteria improves the efficiency of the proxel-based simulation combined with discrete phases. This in turn means that the class of models efficiently analysable by the proxel-based method is extended with this additional check.

The future research prospectives regarding the decision criteria should involve discovering thresholds and heuristics for improving the usability of the phase-proxel simulator.

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